TAIL MEASURES OF STOCHASTIC PROCESSES OR RANDOM FIELDS WITH REGULARLY VARYING TAILS

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Abstract. A new notion called a tail measure is proposed to measure the dependence on extremes of stochastic processes or random fields with regularly varying tails. A tail measure is an infinite-dimensional extension of a family of Radon limiting measures of the regular variation. These measures essentially have structure similar to that of Lévy measures of infinitely divisible processes. A tail measure can be seen, in a sense, to encompass related notions such as extremograms (Davis and Mikosch (2009)) and tail processes (Basrak and Segers (2009)). In addition, focusing on a certain class of stationary infinitely divisible processes, we relate the ergodic properties, described by the positive-null decomposition, to the properties of the probability laws of those processes.

1. Introduction

Heavy-tail analysis typically assumes that a random variable $X$ has an algebraically decaying tail:
\begin{equation}
P(X > x) \sim C x^{-\alpha} L(x), \quad \text{as } x \to \infty,
\end{equation}
where $C > 0$ and $\alpha > 0$ are constants, and $L$ is a slowly varying function; that is, $L(tx)/L(x) \to 1$ as $x \to \infty$ for all $t > 0$. $X$ is then said to have a regularly varying tail with index $\alpha$. If $0 < \alpha < 2$, $X$ has infinite variance, and if $0 < \alpha < 1$, even the mean of $X$ becomes infinite. Heavy-tail analysis applies to the systems governed by a series of extremal events that occur at a non-negligible rate. Indeed, the heavy-tail assumption (1.1) has been applied to diverse fields such as data network analysis, finance, insurance, and natural disasters; for details, see Adler et al. (1998), Beirlant et al. (2004), de Haan and Ferreira (2006), Embrechts et al. (1997), and McNeil et al. (2005).

The heavy-tail assumption (1.1) can be extended to a multivariate form as follows. Let $X$ be a $d$-dimensional random vector. $X$ is said to have a regularly varying tail if there exists a function $H : (0, \infty) \to (0, \infty)$ growing to infinity, and a nonzero Radon measure $\mu$ on $\mathbb{R}^k \setminus \{0\} = [\infty, \infty]^k \setminus \{0\}$ with $\mu(\mathbb{R}^k \setminus \mathbb{R}^k) = 0$, such that
\begin{equation}
H(u) P\left( u^{-1} X \in \cdot \right) \xrightarrow{\text{v}} \mu(\cdot)
\end{equation}

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vaguely in $\mathbb{R}^k \setminus \{0\}$. Various alternative definitions to (1.2) exist. Among the most popular is that there exists a random vector $\Theta$ on a unit sphere $S^{k-1} = \{x \in \mathbb{R}^k : |x| = 1\}$, such that
\[
P(\|X\| > u, X/\|X\| \in \cdot) \Rightarrow x^{-a}P(\Theta \in \cdot)
\]
weakly in $S^{k-1}$. The probability measure $P \circ \Theta^{-1}$ is said to be a spectral measure. For other alternative definitions, we refer to Basrak et al. (2002a) and Resnick (2007).

Researchers are becoming increasingly interested in developing the statistics to measure dependence on extremes of stochastic processes. To this end, Ledford and Tawn (2003) introduced the so-called upper tail dependence coefficient; for a stationary sequence $(X_n, n \geq 0)$ of random variables,
\[
\lambda(n) = \lim_{x \to \infty} P(X_n > x | X_0 > x).
\]
Under the multivariate regular variation condition (1.2), Davis and Mikosch (2009) considered extremograms; for a stationary sequence $(X_n, n \geq 0)$ of $\mathbb{R}^d$-valued random vectors,
\[
\gamma_{AB}(n) = \lim_{u \to \infty} H(u)^{-1}P(u^{-1}X_0 \in A, u^{-1}X_n \in B),
\]
\[
\rho_{AB}(n) = \lim_{u \to \infty} P(u^{-1}X_n \in B | u^{-1}X_0 \in A),
\]
where $H : (0, \infty) \to (0, \infty)$ is a regularly varying function, and both $A$ and $A \times B$ are Borel sets bounded away from zero. Resnick (2004) analyzed the alternative notion of extreme dependence measure. Fasen (2010) provides an elegant review of these types of notions.

However, all of these measures mainly describe dependence on extremes between two vectors only, $X_0$ and $X_n$. We are thus naturally motivated to develop the statistics for describing high-level dependence of the whole process $X$. Let $T$ be a (possibly infinite) parameter space and $X = (X_t, t \in T)$ be a stochastic process or a random field. We further assume that $X$ has regularly varying tails. That is, for all $k \geq 1$ and $t_1, \ldots, t_k \in T$, the random vector $(X_{t_1}, \ldots, X_{t_k})$ has a regularly varying tail with limiting measure $\mu_{t_1 \ldots t_k}$. In this paper, we shall define a cylindrical measure $\nu$ on $\mathbb{R}^T$, termed a tail measure, such that for all $k \geq 1$ and $t_1, \ldots, t_k \in T$,
\[
\nu\left\{ x \in \mathbb{R}^T : (x_{t_1}, \ldots, x_{t_k}) \in B \right\} = \mu_{t_1 \ldots t_k}(B), \quad B \subseteq \mathbb{R}^k \setminus \{0\}.
\]
The measure $\nu$ can be seen to be an infinite-dimensional extension of a family of Radon measures $(\mu_{t_1 \ldots t_k} : t_1, \ldots, t_k \in T, k \geq 1)$.

Basrak and Segers (2009) proposed a related infinite-dimensional object, called a tail process, which contains information on high-level dependence for multivariate time series models. The notion of regular variation in $\mathbb{R}^d$ has been extended to probability laws on non-locally compact metric spaces (e.g. $C[0,1]$; see Hult and Lindskog (2005, 2006). Such an extension, however,
requires that usual vague convergence be replaced by the so-called $\tilde{w}$-convergence ($\tilde{w}$-convergence is detailed in Daley and Vere-Jones (2003)).

This paper is organized as follows. Section 2 provides a rigorous definition of a tail measure as a cylindrical measure $\nu$ given in (1.6). Subsequently, we study several properties of tail measures. We will then make use of the fact that, as an infinite-dimensional measure defined on a big space $\mathbb{R}^T$ ($T$ is an arbitrary parameter space), tail measures have similar structure as (function level) Lévy measures of infinitely divisible processes. In fact, our argument is heavily inspired by ?, which is an instructive lecture note on infinitely divisible processes and is partly based on Maruyama (1970).

Section 3 covers several examples of tail measures for moving averages, independent processes, stochastic volatility processes and GARCH processes. Section 4 reveals the relation between tail measures and other related notions such as extremograms and tail processes. Finally, focusing on stationary infinitely divisible processes of stochastic integral forms, we will investigate the connection between the ergodic theoretical properties of tail measures and those of the probability laws of the processes.

2. Tail Measures and Their Properties

Let $T$ be an arbitrary (possibly infinite) index set and let $(X_t, t \in T)$ be a stochastic process or a random field, e.g., $T = \mathbb{Z}$ for a univariate time series, $T = \mathbb{Z} \times \{1, \ldots, k\}$ for a multivariate time series, and $T = \mathbb{R}^d$ for a random field. It is assumed throughout this paper is that for any parameter space $T$, $(X_t, t \in T)$ has regularly varying tails. More precisely, we suppose that there exists a function $H : (0, \infty) \to (0, \infty)$ growing to infinity such that for all $t_1, \ldots, t_k \in T$, $k \geq 1$, there is a Radon measure $\mu_{t_1 \ldots t_k}$ on $\mathbb{R}^k \setminus \{0\}$ with $\mu(\mathbb{R}^k \setminus \mathbb{R}^k) = 0$, such that as $u \to \infty$,

\[
H(u)P\left( (X_{t_1}, \ldots, X_{t_k}) \in u \cdot \right) \overset{v}{\to} \mu_{t_1 \ldots t_k}(\cdot)
\]

vaguely in $\mathbb{R}^k \setminus \{0\}$. We assume that for at least one $t_0 \in T$, $\mu_{t_0}$ is a nonzero measure. With this assumption, the standard argument regarding regular variation (e.g., Resnick (2007)) shows that $H(x)$ is regularly varying with index $-\alpha$ for some $\alpha > 0$. Furthermore, the Radon measure $\mu_{t_1 \ldots t_k}$, $t_1, \ldots, t_k \in T$ satisfies the homogeneity property: $\mu_{t_1 \ldots t_k}(sA) = s^{-\alpha} \mu_{t_1 \ldots t_k}(A)$ for all Borel sets $A \subseteq \mathbb{R}^k \setminus \{0\}$ and $s > 0$. The existence of the homogeneity exponent $\alpha$ is often emphasized by saying that $(X_t, t \in T)$ has regularly varying tails with index $\alpha$.

Each Radon measure $\mu_{t_1 \ldots t_k}$ can be seen to contain information regarding the high-level dependence of $(X_{t_1}, \ldots, X_{t_k})$. However, merely observing each $\mu_{t_1 \ldots t_k}$ will be insufficient if one hopes to comprehensively capture the extremal behavior of a stochastic process or a random field as a whole. This is particularly true for the relation between the probability laws of finite-dimensional random vectors and the probability law of a stochastic process. We may fail to keep track of the
way a stochastic process evolves dynamically if we focus solely on a family of finite-dimensional random vectors. However, the Kolmogorov extension theorem allows one to clarify the connection between the finite-dimensional random vectors and the corresponding stochastic process. Indeed, given a family of probability laws of these vectors, this theorem guarantees the existence of the corresponding stochastic process. On the other hand, the construction of an infinite-dimensional object that unifies the family of finite-dimensional Radon measures in (2.1) is a nontrivial matter. This is because Radon measures in (2.1) blow up at the origin, whence \( \mu_{t_1 \ldots t_k}(\mathbb{R}^k \setminus \{0\}) = \infty \), and this disallows standard use of Kolmogorov extension theorem.

However, it is still possible to prove the existence of such an infinite-dimensional measure using method suggested by Maruyama (1970). Let \( T \) be an arbitrary parameter space and let \( (Y_t, t \in T) \) be an infinitely divisible process. That is, for all \( k \geq 1 \) and \( t_1, \ldots, t_k \in T \), \( (Y_{t_1}, \ldots, Y_{t_k}) \) forms an infinitely divisible random vector. Specifically, the law of \( (Y_{t_1}, \ldots, Y_{t_k}) \) is identified by a triplet \((\Sigma_F, \rho_F, b_F)\), \( F = \{t_1, \ldots, t_k\} \), where \( \Sigma_F \) is the covariance matrix of the Gaussian part, \( b_F \in \mathbb{R}^F \), and \( \rho_F \) is the Lévy measure. Importantly, the system of Lévy measures \( \{\rho_F : F \subseteq T, \text{finite}\} \) is consistent in the sense that for all finite index sets \( F \subseteq G \subseteq T \),

\[
(2.2) \quad \rho_F(B) = \rho_G\left(\rho_F^{-1}(B \setminus \{0_F\})\right), \quad B \in \mathcal{B}(\mathbb{R}^F),
\]

where \( \rho_G : \mathbb{R}^G \to \mathbb{R}^F \) is the projection (represented by a \( |F| \times |G| \) matrix), and \( 0_F \) is the origin of \( \mathbb{R}^F \). Furthermore, (2.2) implies that every Lévy measure has no mass at the origin, i.e., for every finite \( F \subseteq T \),

\[
(2.3) \quad \rho_F(\{0_F\}) = 0.
\]

Exploiting the structural properties (2.2) (and (2.3)) of finite-dimensional Lévy measures, Maruyama (1970) proves the existence of a big triplet \((\Sigma, \rho, b)\) that characterizes the distribution of the whole process \( Y = (Y_t, t \in T) \). As a result, one can reconstruct each triplet \((\Sigma_F, \rho_F, b_F)\) from \((\Sigma, \rho, b)\) by

\[
\begin{align*}
\Sigma_F &= p_F \Sigma_F p_F^T, \\
\rho_F(B) &= \rho\left(\rho_F^{-1}(B \setminus \{0_F\})\right), \quad B \in \mathcal{B}(\mathbb{R}^F), \\
b_F &= p_F b,
\end{align*}
\]

where \( p_F : \mathbb{R}^T \to \mathbb{R}^F \) is the projection given by \( p_F x = x|_F \), \( x \in \mathbb{R}^T \), and its adjoint \( p_F^T \) satisfies

\[
\left((p_F^T z)_t\right)_t = \begin{cases} z_t & t \in F, \\ 0 & t \in T \setminus F. \end{cases}
\]

The situation for Radon measures defined by (2.1) is analogous to that for finite-dimensional Lévy measures of an infinitely divisible process. Indeed, in Theorem 2.1 below, slightly modified
versions of the Radon measures in (2.1) will be shown to satisfy (2.2) and (2.3). Consequently, the resulting cylindrical measure turns out to be well-defined on the space \( \mathbb{R}^T \).

**Theorem 2.1.** Let \( X = (X_t, t \in T) \) be a stochastic process or a random field, assuming regularly varying tails in the sense of (2.1). Let \( \mathcal{B}(\mathbb{R})^T = \prod_{t \in T} \mathcal{B}(\mathbb{R}_t) \), where \( \mathbb{R}_t = \mathbb{R} \), be the cylindrical \( \sigma \)-field on \( \mathbb{R}^T \). Then a cylindrical measure \( \nu \) on \( (\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T) \), satisfying (i) and (ii) below, uniquely exists. This measure is called a tail measure of \( X \).

(i) : For any finite index set \( F = \{t_1, \ldots, t_k\} \subseteq T \),

\[ \mu_F(A) = \nu(p_F^{-1}(A)) \]

for every Borel set \( A \subseteq \mathbb{R}^F \setminus \{0_F\} \).

(ii) : For every countable \( T_1 \subseteq T \), there exists a countable set \( T_2 \), such that \( T_1 \subseteq T_2 \subseteq T \) and

\[ \nu\left(p^{-1}_{T_1}((0_{T_1})) \right) = \nu\left(p^{-1}_{T_1}((0_{T_1})) \setminus p^{-1}_{T_2}((0_{T_2})) \right) \]

**Remark 2.2.** Condition (ii) in Theorem 2.1 indirectly tells us that a tail measure \( \nu \) has no mass at the origin (note that if \( T \) is uncountable, then the statement \( \nu(\{0_T\}) = 0 \) does not make sense, because \( \{0_T\} \) is not measurable in \( \mathbb{R}^T \)). As an evidence, we can show that if \( T \) is countable, condition (ii) is equivalent to \( \nu(\{0_T\}) = 0 \). To see this, condition (ii) implies

\[ \nu(\{0_T\}) = \nu(p^{-1}_{T_1}((0_{T_1})) \) = \nu\left(p^{-1}_{T_1}((0_{T_1})) \setminus p^{-1}_{T_2}((0_{T_2})) \right) \]

Conversely, if \( \nu(\{0_T\}) = 0 \), then for every countable set \( T_1 \subseteq T \),

\[ \nu\left(p^{-1}_{T_1}((0_{T_1})) \setminus p^{-1}_{T_2}((0_{T_2})) \right) = \nu\left(p^{-1}_{T_1}((0_{T_1})) \right) \]

**Remark 2.3.** If there exists a countable set \( T_0 \subseteq T \) such that \( \nu\left(p^{-1}_{T_0}((0_{T_0})) \right) = 0 \), then condition (ii) follows. To show this, we take an arbitrary countable set \( T_1 \subseteq T \). Define \( T_2 = T_0 \cup T_1 \), which is still countable. Since \( \nu\left(p^{-1}_{T_2}((0_{T_2})) \right) \leq \nu\left(p^{-1}_{T_0}((0_{T_0})) \right) = 0 \),

\[ \nu\left(p^{-1}_{T_2}((0_{T_1})) \setminus p^{-1}_{T_2}((0_{T_2})) \right) = \nu\left(p^{-1}_{T_1}((0_{T_1})) \right) \]

As we will see in Proposition 2.4 below, if one can find such a countable set \( T_0 \subseteq T \), then \( \nu \) turns out to be \( \sigma \)-finite.

**Proof.** First, we identify every \( \mu_F \), with finite \( F \subseteq T \), defined on \( \mathbb{R}^F \setminus \{0_F\} \) by a measure \( \nu_F \) on \( \mathbb{R}^F \) as follows:

\[ \begin{cases} 
\nu_F(\{0_F\}) = 0, \\
\nu_F(A) = \mu_F(A) \quad \text{for any Borel set } A \subseteq \mathbb{R}^F \setminus \{0_F\}. 
\end{cases} \]

We claim that for any finite sets \( F \subseteq G \subseteq T \),

\[ \nu_F(B) = \nu_G\left(p^{-1}_{GF}(B \setminus \{0_F\}) \right), \quad B \in \mathcal{B}(\mathbb{R}^F). \]
We first show (2.4) for every $B \in \mathcal{B}(\mathbb{R}^F)$ with $0_F \notin B$. Fix $G \subseteq T$ and prove this inductively with respect to $\dim(F) \in \{1, \ldots, \dim(G)\}$. Suppose $\dim(F) = 1$. Then, it suffices to show that

$$\nu_F((-\infty, -a] \cup [b, \infty)) = \nu_G(p_{GF}^{-1}((-\infty, -a] \cup [b, \infty)))$$

for every $a > 0$, $b > 0$. We can assume without loss of generality that $(-\infty, -a] \cup [b, \infty)$ and $p_{GF}^{-1}((-\infty, -a] \cup [b, \infty))$ both are continuity sets. Since $(-\infty, -a] \cup [b, \infty)$ is relatively compact on $\mathbb{R}^F \setminus \{0\}$, the Portmanteau theorem for vague convergence (see e.g. Proposition 3.12 in Resnick (1987)) gives

$$\nu_F((-\infty, -a] \cup [b, \infty)) = \mu_F((-\infty, -a] \cup [b, \infty))$$

$$= \lim_{u \to \infty} H(u)P(u^{-1}X_F \in (-\infty, -a] \cup [b, \infty))$$

$$= \lim_{u \to \infty} H(u)P(u^{-1}X_G \in p_{GF}^{-1}((-\infty, -a] \cup [b, \infty))) .$$

Since $p_{GF}^{-1}((-\infty, -a] \cup [b, \infty))$ is relatively compact on $\mathbb{R}^G \setminus \{0_G\}$, one more application of the Portmanteau theorem concludes

$$\lim_{u \to \infty} H(u)P(u^{-1}X_G \in p_{GF}^{-1}((-\infty, -a] \cup [b, \infty)))$$

$$= \mu_G(p_{GF}^{-1}((-\infty, -a] \cup [b, \infty))) = \nu_G(p_{GF}^{-1}((-\infty, -a] \cup [b, \infty))) .$$

Next, suppose that (2.4) is true as long as $1 \leq \dim(F) \leq m < \dim(G)$. We take $\dim(F) = m + 1$ and let $a = (a_1, \ldots, a_{m+1})$ and $b = (b_1, \ldots, b_{m+1})$. We need to show that

$$\nu_F((a, b)^c) = \nu_G(p_{GF}^{-1}((a, b)^c))$$

for every $a, b \in [0, \infty)^F \setminus \{0_F\}$ with $(-a, b)^c$ and $p_{GF}^{-1}((-a, b)^c)$ both continuity sets. If $a_i = 0, b_i > 0$ (or $a_i > 0, b_i = 0$) for some $i \in \{1, \ldots, m + 1\}$, then $0_F \in (-a, b)^c$; therefore, one does not need to consider such cases. If $a_i = b_i = 0$ for some $i \in \{1, \ldots, m + 1\}$, the statement is automatically true by induction hypothesis. Hence, it suffices to check the cases $a_i > 0, b_i > 0$ for all $i \in \{1, \ldots, m + 1\}$. Then, the same argument as applied in the one-dimensional case finishes the proof. To complete (2.4) for any Borel set, consider $B \in \mathcal{B}(\mathbb{R}^F)$ with $0_F \in B$. Since $\nu_F(\{0_F\}) = 0$,

$$\nu_F(B) = \nu_F(B \setminus \{0_F\}) = \nu_G(p_{GF}^{-1}(B \setminus \{0_F\})).$$

We have seen that a family of measures $(\nu_F, F \subseteq T, \text{finite})$, with each $\nu_F$ defined on $(\mathbb{R}^F, \mathcal{B}(\mathbb{R}^F))$, satisfies the same conditions as (2.2) (and hence (2.3)). Now, the Kolmogorov extension-like argument, which was essentially adopted in Proposition 1.1 of Maruyama (1970), proves the existence of a cylindrical measure that fulfills (i) and (ii) in Theorem 2.1.

To prove uniqueness of a tail measure, we suppose that there exists another tail measure $\rho$ on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T)$, such that

$$\nu(p_F^{-1}(B \setminus \{0_F\})) = \rho(p_F^{-1}(B \setminus \{0_F\})), \quad B \in \mathcal{B}(\mathbb{R}^F),$$

(2.5)
for all finite sets \( F \subseteq T \). In the sequel, we will prove that \( \nu = \rho \). Since \( \mathcal{B}(\mathbb{R})^T \) can be expressed as
\[
\mathcal{B}(\mathbb{R})^T = \{ p_S^{-1}(B) : B \in \mathcal{B}(\mathbb{R}^s), S \subseteq T \text{ is a countable set} \},
\]
it is enough to show that
\[
\nu \circ p_S^{-1} = \rho \circ p_S^{-1} \quad \text{for any countable set } S \subseteq T.
\]
By Monotone class theorem, it suffices to check (2.7) for all finite sets \( F \subseteq T \). For every \( B \in \mathcal{B}(\mathbb{R}^F) \),
\[
\nu\left( p_F^{-1}(B) \right) = \nu\left( p_F^{-1}(B \setminus \{0_F\}) \right) + \nu\left( p_F^{-1}(B \cap \{0_F\}) \right)
\]
\[
= \rho\left( p_F^{-1}(B \setminus \{0_F\}) \right) + \nu\left( p_F^{-1}(\{0_F\}) \right) 1_{\{0_F \in B\}}.
\]
Thus, (2.7) will be established if
\[
\nu\left( p_F^{-1}(\{0_F\}) \right) = \rho\left( p_F^{-1}(\{0_F\}) \right)
\]
for all finite \( F \subseteq T \).

By condition (ii) in Theorem 2.1, there is a countable set \( F \subseteq S \subseteq T \), such that
\[
\nu\left( p_F^{-1}(\{0_F\}) \right) = \nu\left( p_F^{-1}(\{0_F\}) \setminus p_S^{-1}(\{0_S\}) \right),
\]
\[
\rho\left( p_F^{-1}(\{0_F\}) \right) = \rho\left( p_F^{-1}(\{0_F\}) \setminus p_S^{-1}(\{0_S\}) \right).
\]
Since \( S \) is countable, there exists a sequence of finite sets \( F \subseteq G_n \uparrow S \) so that
\[
\nu\left( p_F^{-1}(\{0_F\}) \right) = \lim_{n \to \infty} \nu\left( p_F^{-1}(\{0_F\}) \setminus p_G^{-1}(\{0_G\}) \right) = \lim_{n \to \infty} \nu\left( p_G^{-1}(p_G^{-1}(\{0_F\}) \setminus \{0_G\}) \right).
\]
Similarly, we get
\[
\rho\left( p_F^{-1}(\{0_F\}) \right) = \lim_{n \to \infty} \rho\left( p_G^{-1}(p_G^{-1}(\{0_F\}) \setminus \{0_G\}) \right).
\]
Now, (2.5) finishes the proof. \( \square \)

Although a tail measure is not necessarily \( \sigma \)-finite, the next proposition provides a necessary and sufficient condition for it to be \( \sigma \)-finite.

**Proposition 2.4.** Under the assumptions of Theorem 2.1, a tail measure \( \nu \) is \( \sigma \)-finite if and only if there is a countable set \( T_0 \subseteq T \), such that \( \nu\left( p_{T_0}^{-1}(\{0_{T_0}\}) \right) = 0 \).

**Proof.** Suppose, first, that \( \nu \) is \( \sigma \)-finite. There is a sequence \( (A_j) \subseteq \mathcal{B}(\mathbb{R})^T \) with \( \mathbb{R}^T = \bigcup_{j=1}^{\infty} A_j \) and \( \nu(A_j) < \infty \). In view of (2.6), each \( A_j \) can be written as \( A_j = p_{S_j}^{-1}(B_j) \) for some countable \( S_j \subseteq T \) and \( B_j \in \mathcal{B}(\mathbb{R}^{S_j}) \). Define a countable set \( T_1 = \bigcup_{j=1}^{\infty} S_j \).

If \( 0_{S_j} \in B_j \) for some \( j \geq 1 \), then \( p_{T_1}^{-1}(\{0_{T_1}\}) \subseteq p_{S_j}^{-1}(B_j) \) and, hence, \( \nu\left( p_{T_1}^{-1}(\{0_{T_1}\}) \right) \leq \nu(A_j) < \infty \).

From condition (ii) in Theorem 2.1, it follows that
\[
\nu\left( p_{T_1}^{-1}(\{0_{T_1}\}) \setminus p_{T_2}^{-1}(\{0_{T_2}\}) \right) = \nu\left( p_{T_1}^{-1}(\{0_{T_1}\}) \right)
\]
for some countable $T_2$ with $T_1 \subseteq T_2 \subseteq T$.

Now we get
\[
\nu\left(p_{T_2}^{-1}(\{0\})\right) = \nu\left(p_{T_1}^{-1}(\{0\}) \setminus \left(p_{T_1}^{-1}(\{0\}) \setminus p_{T_2}^{-1}(\{0\})\right)\right) = \nu\left(p_{T_1}^{-1}(\{0\})\right) - \nu\left(p_{T_1}^{-1}(\{0\})\right) = 0.
\]

On the contrary, if $0 \notin S_j \in B_j$ for all $j \geq 1$,
\[
\nu\left(p_{T_1}^{-1}(\{0\})\right) \leq \sum_{j=1}^{\infty} \nu\left(p_{T_1}^{-1}(\{0\}) \cap p_{S_j}^{-1}(B_j)\right) = \sum_{j=1}^{\infty} \nu(\emptyset) = 0.
\]

Conversely, assume that $\nu\left(p_{T_0}^{-1}(\{0\})\right) = 0$ for some countable $T_0 \subseteq T$. We can express $\mathbb{R}^T$ by
\[
\mathbb{R}^T = \bigcup_{t \in T_0} \bigcup_{n=1}^{\infty} \left(p_{T_0}^{-1}(\{0\}) \cup \{x \in \mathbb{R}^T : |x_1| > n^{-1}\}\right).
\]

Since
\[
\nu\left(p_{T_0}^{-1}(\{0\}) \cup \{x \in \mathbb{R}^T : |x_1| > n^{-1}\}\right) = \nu(x \in \mathbb{R}^T : |x_1| > n^{-1}) = \mu(y : |y| > n^{-1}) < \infty,
\]
$\nu$ is $\sigma$-finite.

The next proposition describes the homogeneity property of a tail measure. It can be proved directly by the homogeneity of finite-dimensional Radon measures in (2.1). So, we only presents the result.

**Proposition 2.5.** Under the assumptions of Theorem 2.1, a tail measure $\nu$ has the homogeneity property. That is, there exists an $\alpha > 0$ such that
\[
\nu(cA) = c^{-\alpha} \nu(A) \quad \text{for all } A \in \mathcal{B}(\mathbb{R})^T, \ c > 0.
\]

### 3. Examples

This section presents the tail measures of several stochastic processes; moving averages, independent processes, stochastic volatility processes, and GARCH processes.

**Example 3.1.** Let $T = \mathbb{Z}$ or $\mathbb{R}$, and $(X_t, t \in T)$ be a stochastic process with integral representation
\[
X_t = \int_E f_t(x) dM(x), \quad t \in T,
\]
where $M$ is an independently scattered infinitely divisible random measure on a measurable space $(E, \mathcal{E})$ with $\sigma$-finite control measure $m$ and a local Lévy measure $\rho(s, \cdot)$, $s \in E$. The functions $f_t$ are deterministic functions of the form $f_t(x) = f \circ \psi_t(x)$, $x \in E$, $t \in T$, where $f : E \to \mathbb{R}$ is a measurable function, and $\psi_t : E \to E$, $t \in T$ is a family of measurable maps. Rajput and Rosiński (1989) describes a condition under which $X_t$ is well-defined, which will be assumed throughout
this example. Then \((X_t, t \in T)\) is, automatically, an well-defined infinitely divisible process. The function level Lévy measure of \((X_t, t \in T)\) is given by \((\rho \times m) \circ h^{-1}\), where \(h(x, s) = xf(s), x \in \mathbb{R}, s \in E\). Furthermore, we suppose the following conditions:

- There exists a measurable and regularly varying function \(H : (0, \infty) \to (0, \infty)\) of index \(-\alpha\) for some \(\alpha > 0\). There also exist measurable functions \(w_{\pm} : E \to [0, \infty)\) such that for every \(s \in E\),

\[
\lim_{u \to \infty} \frac{\rho(s, (u, \infty))}{H(u)} = w_+(s) \quad \text{and} \quad \lim_{u \to \infty} \frac{\rho(s, (-\infty, -u))}{H(u)} = w_-(s).
\]

- The convergence above is uniform: there exists \(u_0 > 0\) with

\[
\sup_{u \geq u_0} \frac{\rho(s, (u, \infty))}{H(u)} \leq 2w_+(s) \quad \text{and} \quad \sup_{u \geq u_0} \frac{\rho(s, (-\infty, -u))}{H(u)} \leq 2w_-(s)
\]

for all \(s \in E\).

- \(f : E \to \mathbb{R}\) is bounded on \(E\) and, for some \(\xi \in (0, \alpha)\),

\[
\int_E w_{\pm}(s)|f_t(s)|^{\alpha-\xi}m(ds) < \infty
\]

for all \(t \in T\).

Then, one can show that \((X_t, t \in T)\) has regularly varying tails with index \(\alpha\) and the tail measure of \((X_t, t \in T)\) is given by

\[\nu = (\rho_\alpha \times m) \circ h^{-1},\]

where

\[
\rho_\alpha(s, dx) = w_+(s)\frac{\alpha}{x^{1+\alpha}}1_{\{x>0\}}dx + w_-(s)\frac{\alpha}{|x|^{1+\alpha}}1_{\{x<0\}}dx.
\]

To show this, we only have to prove that as \(u \to \infty\),

\[H(u)^{-1}P((X_{t_1}, \ldots, X_{t_k}) \in u \cdot) \overset{\text{w}}{\to} (\rho_\alpha \times m)\left\{(x, s) : (xf_{t_1}(s), \ldots, xf_{t_k}(s)) \in \cdot \right\}\]

equivalently in \(\mathbb{R}^k \setminus \{0\}\) for all \(t_1, \ldots, t_k \in T, k \geq 1\).

Equivalently, we need prove that for all \(a_i > 0\) and \(e_i \in \{-1, 1\}, i = 1, \ldots, k\),

\[H(u)^{-1}P(e_iX_{t_i} > a_iu, \ i = 1, \ldots, k) \to (\rho_\alpha \times m)\left\{(x, s) : xe_i f_{t_i}(s) > a_i, \ i = 1, \ldots, k \right\}.
\]

The tail behavior of the probability law of \((X_t, t \in T)\) is known to coincide with that of the function level Lévy measure of \((X_t, t \in T)\). See Theorem 2.1 in Rosiński and Samorodnitsky (1993). Thus, (3.6) is equivalent to

\[H(u)^{-1}(\rho \times m)\left\{(x, s) : xe_i f_{t_i}(s) > a_iu, \ i = 1, \ldots, k \right\} \to (\rho_\alpha \times m)\left\{(x, s) : xe_i f_{t_i}(s) > a_i, \ i = 1, \ldots, k \right\}.
\]
The left hand side of (3.7) is equal to
\[
\int_{\{e_i f_i(s) > 0, \ i = 1, \ldots, k\}} H(u)^{-1} \rho\left(s, (u \max_{1 \leq i \leq k} a_i |f_{i}(s)|^{-1}, \infty)\right) m(ds)
+ \int_{\{e_i f_i(s) < 0, \ i = 1, \ldots, k\}} H(u)^{-1} \rho\left(s, (-\infty, -u \max_{1 \leq i \leq k} a_i |f_{i}(s)|^{-1})\right) m(ds).
\]

On the other hand, the right hand side of (3.7) is equal to
\[
\int_{\{e_i f_i(s) > 0, \ i = 1, \ldots, k\}} w_+(s) \min_{1 \leq i \leq k} \left(\frac{|f_{i}(s)|}{a_i}\right)^{\alpha} m(ds) + \int_{\{e_i f_i(s) < 0, \ i = 1, \ldots, k\}} w_-(s) \min_{1 \leq i \leq k} \left(\frac{|f_{i}(s)|}{a_i}\right)^{\alpha} m(ds).
\]

Due to their symmetric structure, it suffices to check the convergence of the integral defined on \(\{e_i f_i(s) > 0, \ i = 1, \ldots, k\}\). By condition (3.2),
\[
H(u)^{-1} \rho\left(s, (u \max_{1 \leq i \leq k} a_i |f_{i}(s)|^{-1}, \infty)\right) \rightarrow w_+(s) \min_{1 \leq i \leq k} \left(\frac{|f_{i}(s)|}{a_i}\right)^{\alpha}
\]
for every \(s \in E\). Therefore, we only need to justify taking the limit inside. In order to apply the dominated convergence theorem, we must find a nonnegative function \(K \in L^1(E, m)\) such that
\[
H(u)^{-1} \rho\left(s, (u \max_{1 \leq i \leq k} a_i |f_{i}(s)|^{-1}, \infty)\right) \leq K(s)
\]
for every \(s \in E\) and sufficiently large \(u > 0\).

In view of uniformity condition (3.3), for all \(u \geq u_0 \sup_{s \in E} |f(s)| / \max_{1 \leq i \leq k} a_i\) (the right hand side is finite, since \(f\) is bounded),
\[
H(u)^{-1} \rho\left(s, (u \max_{1 \leq i \leq k} a_i |f_{i}(s)|^{-1}, \infty)\right) \leq 2w_+(s) \frac{H\left(u \max_{1 \leq i \leq k} a_i |f_{i}(s)|^{-1}\right)}{H(u)}.
\]

The Potter bounds (see e.g. Proposition 0.8 in Resnick (1987)) provide, for some \(C_i > 0, \ i = 1, 2,\)
\[
\frac{H\left(u \max_{1 \leq i \leq k} a_i |f_{i}(s)|^{-1}\right)}{H(u)} \leq C_1 \left(\max_{1 \leq i \leq k} \frac{a_i}{|f_{i}(s)|}\right)^{-\alpha + \xi} \leq C_2 \sum_{i=1}^{k} |f_{i}(s)|^{a - \xi}
\]
for all \(u\) large enough. Now, because of (3.4), an appropriate \(L^1(E, m)\)-upper bound \(K(\cdot)\) is easy to take and the proof is complete.

**Example 3.2.** We will consider, once again, the process (3.1), but here, a local Lévy measure \(\rho\) is independent of \(s \in E\). We assume that \(\rho\) has a balanced regularly varying tail: for some \(\alpha > 0\), \(\rho(x: |x| > \cdot)\) is regularly varying with index \(-\alpha\), and
\[
\frac{\rho(y, \infty)}{\rho(x: |x| > y)} \rightarrow p, \quad \frac{\rho(-\infty, -y)}{\rho(x: |x| > y)} \rightarrow q
\]
as \(y \rightarrow \infty\), where \(0 \leq p, q \leq 1\) with \(p + q = 1\).
In this example, we remove boundedness assumption of $f$, and instead, the following integrability condition is assumed: there exists $0 < \beta \leq 2$ such that

$$\int_E |f_t(s)|^{\alpha - \xi} \vee |f_t(s)|^\beta m(ds) < \infty \quad \text{for some } \alpha < \beta - \alpha \quad \text{if } 0 < \alpha < \beta,$$

$$\int_E |f_t(s)|^{\alpha - \xi} \vee |f_t(s)|^{\alpha + \xi} m(ds) < \infty \quad \text{for some } \alpha < \beta \quad \text{if } \alpha \geq \beta,$$

for all $t \in T$. If $\beta \neq 2$, the lower tail behavior of $\rho$ has to be specified explicitly, that is,

$$y^\beta \rho(x : |x| > y) \to 0 \quad \text{as } y \downarrow 0.$$

Under these assumptions, $(X_t, t \in T)$ has, once again, regularly varying tails with index $\alpha$ and its tail measure is given by

$$\nu = (\rho_* \times m) \circ h^{-1},$$

where $h$ is the same function as before, and

$$\rho_*(dx) = p \frac{\alpha}{x^{1+\alpha}} 1_{\{x > 0\}} dx + q \frac{\alpha}{\min \{x^{1+\alpha} \}} 1_{\{x < 0\}} dx.$$

For the proof, let $H(u) = \rho(x : |x| > u)$. By the same argument as Example 3.1, it suffices to verify that, for all $t_1, \ldots, t_k \in T$, $k \geq 1$, $a_i > 0$ and $e_i \in \{-1,1\}$, $i = 1, \ldots, k$,

$$\int_{\{e_i f_{t_i}(s) > 0, i = 1, \ldots, k\}} H(u)^{-1} \rho \left( u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1}, \infty \right) m(ds) \to \int_{\{e_i f_{t_i}(s) > 0, i = 1, \ldots, k\}} p \min_{1 \leq i \leq k} \left( \frac{|f_{t_i}(s)|}{a_i} \right)^\alpha m(ds),$$

$$\int_{\{e_i f_{t_i}(s) < 0, i = 1, \ldots, k\}} H(u)^{-1} \rho \left( -\infty, -u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1} \right) m(ds) \to \int_{\{e_i f_{t_i}(s) < 0, i = 1, \ldots, k\}} q \min_{1 \leq i \leq k} \left( \frac{|f_{t_i}(s)|}{a_i} \right)^\alpha m(ds).$$

By regular variation of $H(u)$, we have, as $u \to \infty$,

$$H(u)^{-1} \rho \left( u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1}, \infty \right) \to p \min_{1 \leq i \leq k} \left( \frac{|f_{t_i}(s)|}{a_i} \right)^\alpha,$$

$$H(u)^{-1} \rho \left( -\infty, -u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1} \right) \to q \min_{1 \leq i \leq k} \left( \frac{|f_{t_i}(s)|}{a_i} \right)^\alpha,$$

for any $s \in E$. It remains to find a measurable function $K \in L^1(E, m)$, such that

$$H(u)^{-1} \rho(x : |x| > u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1}) \leq K(s)$$

for any $s \in E$ and sufficiently large $u > 0$.

We see from the Potter bounds that, for some $C_i > 0$, $i = 1, 2$,

$$\frac{\rho(x : |x| > u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1})}{\rho(x : |x| > u)} \leq C_1 \left( \max_{1 \leq i \leq k} \frac{a_i}{|f_{t_i}(s)|} \right)^{-\alpha + \xi} + \left( \max_{1 \leq i \leq k} \frac{a_i}{|f_{t_i}(s)|} \right)^{-\alpha - \xi} \leq C_2 \sum_{i=1}^k \left( |f_{t_i}(s)|^{\alpha - \xi} + |f_{t_i}(s)|^{\alpha + \xi} \right).$$
for all $u$ large enough.

Since $y^\beta \rho(x : |x| > y) \to 0$ as $y \downarrow 0$, there exists $C_3 > 0$ with $\rho(x : |x| > y) < C_3 y^{-\beta}$ for all $0 < y \leq 1$. Thus, we see that

$$
\frac{\rho(x : |x| > u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1})}{\rho(x : |x| > u)} \leq \frac{C_4}{u^\beta \rho(x : |x| > u)} \sum_{i=1}^{k} |f_{t_i}(s)|^\beta \mathbf{1}\left( u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1} \leq 1 \right)
$$

for some $C_4 > 0$. If $0 < \alpha < \beta$, then $u^\beta \rho(x : |x| > u) \to \infty$ as $u \to \infty$ and, hence,

$$
\frac{\rho(x : |x| > u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1})}{\rho(x : |x| > u)} \mathbf{1}\left( u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1} \leq 1 \right) \leq C_4 \sum_{i=1}^{k} |f_{t_i}(s)|^\beta
$$

for all $u$ large enough.

On the contrary, if $\alpha \geq \beta$, there is $C_5 > 0$ such that

$$
\frac{1}{u^\beta \rho(x : |x| > u)} \leq C_5 u^{\alpha-\beta+\xi}
$$

for all $u$ large enough. Therefore, for some $C_6 > 0$,

$$
\frac{\rho(x : |x| > u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1})}{\rho(x : |x| > u)} \mathbf{1}\left( u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1} \leq 1 \right) \leq C_4 C_5 u^{\alpha-\beta+\xi} \sum_{i=1}^{k} |f_{t_i}(s)|^\beta \mathbf{1}\left( u \max_{1 \leq i \leq k} a_i |f_{t_i}(s)|^{-1} \leq 1 \right) \leq C_6 \sum_{i=1}^{k} |f_{t_i}(s)|^{\alpha+\xi}.
$$

In either case, we have found an $L^1(E, m)$-function $K(\cdot)$ as desired.

**Example 3.3.** Let $T$ be an arbitrary set and $\mathbf{X} = (X_{t_i}, t \in T)$ be an independent process (i.e., for all $t_1, \ldots, t_k \in T$, $X_{t_1}, \ldots, X_{t_k}$ are independent). Suppose that there is a regularly varying function $H : (0, \infty) \to (0, \infty)$ with index $-\alpha$ for some $\alpha > 0$, such that for every $t \in T$, there is a Radon measure $\mu_t$ on $\mathbb{R} \setminus \{0\}$ with $\mu(\mathbb{R} \setminus \mathbb{R}) = 0$, such that, as $u \to \infty$,

$$
H(u)^{-1} P(X_t \in u \cdot) \xrightarrow{\mu} \mu_t(\cdot)
$$

guously in $\mathbb{R} \setminus \{0\}$. Assume that at least one Radon measure $\mu_{t_0}$, $t_0 \in T$, is a nonzero measure. Then the tail measure of $\mathbf{X}$ is given by

$$
\nu(A) = \sum_{t \in T} \nu_t(A), \quad A \in \mathcal{B}(\mathbb{R})^T,
$$

where

$$
\nu_t(A) = \mu_t((1, \infty)) \int_0^\infty \alpha y^{-(\alpha+1)} \mathbf{1}_{\{y e(t) \in A\}} dy + \mu_t((-\infty, -1)) \int_{-\infty}^0 \alpha |y|^{-(\alpha+1)} \mathbf{1}_{\{y e(t) \in A\}} dy,
$$

$e(t) \in \mathbb{R}^T$ with $e(t)_s = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{otherwise.} \end{cases}$
To see this, the multivariate regular variation of \((X_{t_1}, \ldots, X_{t_k})\) is derived from (3.12) and independence of \((X_{t_1}, \ldots, X_{t_k})\) (see e.g. Lemma 7.2 in Resnick (2007)):

\[
H(u)^{-1}P\left((X_{t_1}, \ldots, X_{t_k}) \in u \cdot \right) \xrightarrow{u \to \infty} \sum_{j=1}^{k} (\epsilon_0 \times \cdots \times \mu_{t_j} \times \cdots \times \epsilon_0)
\]

generally in \(\mathbb{R}^k \setminus \{0\}\). Here

\[
\epsilon_0(A) = \begin{cases} 1 & \text{if } 0 \in A, \\ 0 & \text{otherwise.} \end{cases}
\]

It is easy to verify that, with \(F = \{t_1, \ldots, t_k\}\),

\[
\nu_t \circ p_F^{-1} = \begin{cases} \epsilon_0 \times \cdots \times \mu_{t_j} \times \cdots \times \epsilon_0, & j = 1, \ldots, k, \\ 0 & \text{if } t \notin F. \end{cases}
\]

Therefore,

\[
\sum_{t \in T} \nu_t \circ p_F^{-1} = \sum_{j=1}^{k} (\epsilon_0 \times \cdots \times \mu_{t_j} \times \cdots \times \epsilon_0).
\]

Since the choice of a finite index set \(F\) is arbitrary, the tail measure of \(X\) turns out to be \(\nu = \sum_{t \in T} \nu_t\).

**Example 3.4.** Let \((X_n, n = 1, 2, \ldots)\) be a simple stochastic volatility process of the form

\[
X_n = \sigma_n Z_n, \quad n = 1, 2, \ldots
\]

where \((\sigma_n)\) is a nonnegative stationary sequence, representing volatility. \((Z_n)\) is a sequence of i.i.d. random variables and is independent of \((\sigma_n)\), and \((Z_n)\) has a regularly varying tail with index \(\alpha\). Let \(\mu\) denote the limiting Radon measure of the regular variation for \((Z_n)\). Assume that volatility sequence \((\sigma_n)\) has a significantly lighter tail than that of \((Z_n)\); that is, \((\sigma_n)\) has finite \((\alpha + \epsilon)\)th moment for some \(\epsilon > 0\). Then, their product \(X = (X_n, n = 1, 2, \ldots)\) becomes a stationary sequence with regularly varying tail of the same index \(\alpha\), and the tail measure of \(X\) is given by

\[
\nu(A) = E(\sigma^n) \sum_{j=1}^{\infty} \nu_j(A)
\]

for a Borel set \(A\), where

\[
\nu_j(A) = \mu((1, \infty)) \int_{0}^{\infty} \alpha y^{-(\alpha+1)} 1_{\{y \in A\}} dy + \mu((-\infty, -1)) \int_{-\infty}^{0} \alpha |y|^{-(\alpha+1)} 1_{\{y \in A\}} dy.
\]

To see this, since \(E\sigma^{\alpha+\epsilon} < \infty\), the multivariate Breiman’s theorem (see e.g. Basrak et al. (2002b)) yields

\[
H(u)^{-1}P\left((\sigma_1 Z_1, \ldots, \sigma_k Z_k) \in u \cdot \right) \xrightarrow{u \to \infty} \sum_{j=1}^{k} E\left[ (\epsilon_0 \times \cdots \times \mu \times \cdots \times \epsilon_0) \left\{ x : (\sigma_1 x_1, \ldots, \sigma_k x_k) \in \cdot \right\} \right]
\]
vaguely in $\mathbb{R}^k \setminus \{0\}$ for every $k \geq 1$. Because of the stationarity of $(\sigma_n)$ and the homogeneity property of $\mu$,
\[
\sum_{j=1}^k E \left[ (\epsilon_0 \times \cdots \times \mu \times \cdots \times \epsilon_0) \{ x : (\sigma_1 x_1, \ldots, \sigma_k x_k) \in \cdot \} \right] = E(\sigma^n) \sum_{j=1}^k (\epsilon_0 \times \cdots \times \mu \times \cdots \times \epsilon_0).
\]
The same argument as in Example 3.3 establishes
\[
\sum_{j=1}^k (\epsilon_0 \times \cdots \times \mu \times \cdots \times \epsilon_0) = \infty \sum_{j=1}^{\nu_j \circ P_{\{1 \ldots k\}}^{-1}},
\]
which finishes the proof.

If $Z_n = Z_1$ for all $n = 1, 2, \ldots$, then $Z_1$ works as a common heavy-tailed component for $(X_n)$, which means that $(X_n)$ is expected to exhibit a longer memory than the previous i.i.d. setup. In this case, the tail measure of $X$ can be simply written as
\[
E\mu \{ x : x(\sigma_1, \sigma_2, \ldots)^' \in \cdot \}.
\]

Example 3.5. This example considers the tail measure of GARCH processes. Regular variation of GARCH processes was rigorously discussed by Basrak et al. (2002b) from the viewpoint of stochastic recurrence equations. A nice review on heavy-tailed GARCH processes, including continuous-time models, is provided by Fasen (2010). In order to calculate the tail measure explicitly, we will concentrate on GARCH(1,1) processes and the argument is, to some extent, parallel to that of Davis and Mikosch (2009). Specifically, we will consider the following GARCH(1,1) process:

\begin{align}
X_n & = \sigma_n Z_n, \quad n \in \mathbb{N}, \\
\sigma_n^2 & = \alpha_0 + \alpha_1 X_{n-1}^2 + \beta_1 \sigma_{n-1}^2,
\end{align}

where $\alpha_0$, $\alpha_1$ and $\beta_1$ are positive constants, $\sigma_0$ is a nonnegative random variable, and $(Z_n)$ are i.i.d. symmetric random variables with unit variance. Let $A_n = \alpha_1 Z_{n-1}^2 + \beta_1$, and suppose that $E \log A = E \log(\alpha_1 Z^2 + \beta_1) < 0$. Under such a circumstance, there exists a stationary solution $(X_n, \sigma_n)$ to stochastic equations (3.13) and (3.14); see Babillot et al. (1997) for more details. Assume, additionally, that the law of $\log A$ is nonarithmetic, $P(A > 1) > 0$ and there exists $1 < h_0 \leq \infty$, such that
\[
E A^h < \infty \quad \text{for all} \ h < h_0 \quad \text{and} \quad E A^{h_0} = \infty.
\]
Then, some constant $\alpha > 0$ satisfies $E A^{\alpha/2} = 1$ and, further, $(\sigma_n)$ is regularly varying with index $\alpha$ (see Mikosch and Stàricà (2000) for a detailed proof). We denote by $\mu$ the limiting Radon measure for the regular variation of $\sigma_n$; namely, there is a function $H : (0, \infty) \to (0, \infty)$ such that
\[
H(u)^{-1} P(\sigma_n \in u \cdot) \overset{v}{\rightarrow} \mu(\cdot)
\]
Davis and Mikosch (2009) gives a useful approximation of \((X)\) extreme dependence measure defined by

\[
E \mu \left\{ x : x(Z_0, Z_1 \sqrt{A_1}, Z_2 \sqrt{A_1 A_2}, \ldots) \in \cdot \right\}.
\]

For the proof, fix \((a_0, \ldots, a_k) \in [0, \infty)^{k+1} \setminus \{0\} \) and \(e_i \in \{-1, 1\}, i = 0, \ldots, k\). Lemma 2.1 in Davis and Mikosch (2009) gives a useful approximation of \((X_0, \ldots, X_k)\):

\[
(X_0, X_1, \ldots, X_k) = \sigma_0(Z_0, Z_1 \sqrt{A_1}, \ldots, Z_k \sqrt{A_1 A_2 \cdots A_k}) + \mathbb{R},
\]

where \(H(u)^{-1}P(\|\mathbf{R}\| > u\epsilon) \to 0\), as \(u \to \infty\), for every \(\epsilon > 0\).

Thus, as \(u \to \infty\), we have

\[
H(u)^{-1}P(e_i X_i > u a_i, \ i = 0, \ldots, k)
\]

\[
\sim H(u)^{-1}P(\sigma_0 e_0 Z_0 > u a_0, \sigma_0 e_1 Z_1 \sqrt{A_1} > u a_1, \ldots, \sigma_0 e_k Z_k \sqrt{A_1 \cdots A_k} > u a_k).
\]

Since \(E(Z_i \sqrt{A_1 \cdots A_i})^{a+\epsilon} < \infty\) for \(\epsilon\) small enough, an application of the multivariate Breiman’s theorem yields

\[
H(u)^{-1}P(\sigma_0 e_0 Z_0 > u a_0, \sigma_0 e_1 Z_1 \sqrt{A_1} > u a_1, \ldots, \sigma_0 e_k Z_k \sqrt{A_1 \cdots A_k} > u a_k)
\]

\[
\to E \mu \left\{ x : x e_0 Z_0 > a_0, \ x e_1 Z_1 \sqrt{A_1} > a_1, \ldots, xe_k Z_k \sqrt{A_1 \cdots A_k} > a_k \right\}
\]
as required.

4. Connection between Tail Measures and Other Related Notions

This section investigates the relation between tail measures and their related notions. Consequently, the tail measure turns out be a more comprehensive notion than those alternatives.

First of all, for a stationary sequence \((X_n, n \geq 0)\), the relation between the tail measure \(\nu\) and upper tail dependence coefficient (1.3) is clearly described by

\[
\lambda(n) = \frac{\nu\{x \in \mathbb{R}^N : \min(x_0, x_n) > 1\}}{\nu\{x \in \mathbb{R}^N : x_0 > 1\}}.
\]

Let \((X_t, t \in T)\), \(T = \mathbb{R}\) or \(\mathbb{Z}\) be a stationary process with tail measure \(\nu\). Fasen (2010) studied extreme dependence measure defined by

\[
\chi_{(t_1, \ldots, t_d)}(y_1, \ldots, y_d) = \lim_{u \to \infty} H(u)^{-1}P(X_{t_1} > uy_1, \ldots, X_{t_d} > uy_d),
\]

\[
\overline{\chi}_{(t_1, \ldots, t_d)}(y_1, \ldots, y_d) = \lim_{u \to \infty} H(u)^{-1}P(X_{t_1} > uy_1 \text{ or} \ldots \text{ or } X_{t_d} > uy_d)
\]

for some regularly varying function \(H : (0, \infty) \to (0, \infty)\). One can obtain an obvious relation

\[
\chi_{(t_1, \ldots, t_d)}(y_1, \ldots, y_d) = \nu\{x \in \mathbb{R}^T : x_{t_j} > y_j, \ j = 1, \ldots, d\}.
\]
The connection between $\nu$ and $\chi$ can also be formulated easily by the inclusion-exclusion property. Let $(X_n, n \geq 0)$ be a stationary process in $\mathbb{R}^d$. The tail measure $\nu$ then relates to extremogram (1.4) and (1.5) in such a way that

$$
\gamma_{AB}(n) = \nu\{ x \in (\mathbb{R}^d)^N : x_0 \in A, x_n \in B \},
$$

$$
\rho_{AB}(n) = \frac{\nu\{ x \in (\mathbb{R}^d)^N : x_0 \in A, x_n \in B \}}{\nu\{ x \in (\mathbb{R}^d)^N : x_0 \in A \}},
$$

where $x_i = (x_{i,1}, \ldots, x_{i,d})$, $i \geq 0$ and both $A$ and $A \times B$ are Borel sets bounded away from zero.

In relation to the examples in the preceding section, Fasen (2010) calculated the upper tail dependence coefficient and the extreme dependence measure of the process in Example 3.5. The extremograms for the processes in Examples 3.4 and 3.5 are provided by Davis and Mikosch (2009). Moreover, it is not difficult to calculate these quantities for the infinitely divisible processes in Examples 3.1 and 3.2.

We will consider a multivariate stationary time-series $X = (X_n, n \in \mathbb{Z})$ in $\mathbb{R}^d$ with regularly varying tails of index $\alpha > 0$. Basrak and Segers (2009) defined a limiting process $Y = (Y_n, n \in \mathbb{Z})$ in $\mathbb{R}^d$, called tail process, by

$$
P\left( (X_m, \ldots, X_n) \in u \cdot \|X_0\| > u \right) \to P\left( (Y_m, \ldots, Y_n) \in \cdot \right)
$$

weakly in $\mathbb{R}^{d(n-m+1)}$ for all $m, n \in \mathbb{Z}$ with $m \leq n$. Here $\| \cdot \|$ is an arbitrary norm on $\mathbb{R}^d$. On the other hand, the tail measure $\nu$ of $X$ satisfies

$$
P\left( (X_m, \ldots, X_n) \in u \cdot \|X_0\| > u \right) \to \frac{\nu\{ x \in (\mathbb{R}^d)^\mathbb{Z} : (x_m, \ldots, x_n) \in \cdot, \|x_0\| > 1 \}}{\nu\{ x \in (\mathbb{R}^d)^\mathbb{Z} : \|x_0\| > 1 \}}.
$$

Therefore, we conclude

$$
P(Y \in \cdot) = \frac{\nu\{ x \in (\mathbb{R}^d)^\mathbb{Z} : \|x_0\| > 1 \}}{\nu\{ x \in (\mathbb{R}^d)^\mathbb{Z} : \|x_0\| > 1 \}}.
$$

Basrak and Segers (2009) also defined the spectral process of $X$ by $\Theta_n = Y_n/\|Y_0\|$, $n \in \mathbb{Z}$. Because of their Corollary 3.2, we find that $\Theta = (\Theta_n, n \in \mathbb{Z})$ fulfills

$$
P(\Theta \in \cdot) = \frac{\nu\{ x \in (\mathbb{R}^d)^\mathbb{Z} : x/\|x_0\| \in \cdot, \|x_0\| > 1 \}}{\nu\{ x \in (\mathbb{R}^d)^\mathbb{Z} : \|x_0\| > 1 \}}.
$$

The two most important properties of the tail process and the spectral process of $X$ are given in Theorem 3.1 in Basrak and Segers (2009). For instance, statement (iii) of that theorem says that for all $i, m, n \in \mathbb{Z}$ with $m \leq 0 \leq n$ and for all bounded and continuous $f : (\mathbb{R}^d)^{n-m+1} \to \mathbb{R}$,

$$
E\left[ f(\Theta_{m-i}, \ldots, \Theta_{n-i}) \right] = E\left[ f\left( \frac{\Theta_m}{\|\Theta_i\|}, \ldots, \frac{\Theta_n}{\|\Theta_i\|} \right) \|\Theta_i\|^\alpha \right].
$$
Exploiting some nice properties of tail measures, we can provide a more natural alternative proof of (4.1). First, recall that the tail measure \( \nu \) possesses homogeneity property as mentioned in Proposition 2.5. The second nice property is that due to the stationarity of \( X \), \( \nu \) is shift invariant, that is,

\[
\nu \circ \phi_n^{-1} = \nu \quad \text{for every } n \in \mathbb{Z},
\]

where \( \phi_n : (\mathbb{R}^d)^\mathbb{Z} \to (\mathbb{R}^d)^\mathbb{Z} \) is the shift operator defined by

\[
\phi_n(\ldots, x_{-1}, x_0, x_1, \ldots) = (\ldots, x_{n-1}, x_n, x_{n+1}, \ldots).
\]

For the proof of (4.1), suppose for notational ease that \( \nu \{ x \in (\mathbb{R}^d)^\mathbb{Z} : \| x_0 \| > 1 \} = 1 \). Using the identity

\[
\frac{1}{\| x_- \|^\alpha} \int_0^{\| x_- \|} \alpha u^{\alpha-1} \, du = 1,
\]

we write

\[
E \left[ f(\Theta_{m-1}, \ldots, \Theta_{n-1}) \right] = \int_0^\infty \int_{(\mathbb{R}^d)^\mathbb{Z}} f \left( \frac{x_{m-i}}{\| x_0 \|}, \ldots, \frac{x_{n-i}}{\| x_0 \|} \right) \frac{\alpha u^{\alpha-1}}{\| x_- \|^\alpha} 1_{\{\| x_- \| > u, \| x_0 \| > 1\}} \, \nu(dx) \, du.
\]

By virtue of the shift invariance and the homogeneity property of \( \nu \),

\[
\int_0^\infty \int_{(\mathbb{R}^d)^\mathbb{Z}} f \left( \frac{x_{m-i}}{\| x_0 \|}, \ldots, \frac{x_{n-i}}{\| x_0 \|} \right) \frac{\alpha u^{\alpha-1}}{\| x_- \|^\alpha} 1_{\{\| x_- \| > u, \| x_0 \| > 1\}} \, \nu(dx) \, du
\]

\[
= \int_0^\infty \int_{(\mathbb{R}^d)^\mathbb{Z}} f \left( \frac{x_{m-i}}{\| x_0 \|}, \ldots, \frac{x_{n-i}}{\| x_0 \|} \right) \frac{\alpha u^{\alpha-1}}{\| x_- \|^\alpha} 1_{\{\| x_- \| > 1, \| x_0 \| > u^{-1}\}} \, \nu(dx) \, du
\]

\[
= \int_{(\mathbb{R}^d)^\mathbb{Z}} f \left( \frac{x_m}{\| x \|}, \ldots, \frac{x_n}{\| x \|} \right) \frac{\| x \|}{\| x_- \|^\alpha} 1_{\{\| x_0 \| > 1\}} \, \nu(dx)
\]

\[
= E \left[ f \left( \frac{\Theta_m}{\| \Theta \|}, \ldots, \frac{\Theta_n}{\| \Theta \|} \right) \right].
\]

Notice that a similar argument can prove statement (ii) of Theorem 3.1 in Basrak and Segers (2009) as well.

5. Application: Ergodic Theoretical Properties of Tail Measures and Those of Probability Laws of \((X_t, t \in T)\)

In this section, we will always consider a stationary process \( X = (X_t, t \in T) \) with \( T = \mathbb{Z} \) or \( \mathbb{R} \), assuming that \( X \) has regularly varying tails. Let \( \nu \) be the tail measure of \( X \). As pointed out in the preceding section, \( \nu \) is shift invariant:

\[
\nu \circ \phi_t^{-1} = \nu \quad \text{for every } t \in T,
\]
where \( \phi_t : \mathbb{R}^T \to \mathbb{R}^T \) is defined by \( \phi_t(x) = x_{t-} \), \( x \in \mathbb{R}^T \). Thus, we are motivated to study the properties of the tail measure from ergodic-theoretical viewpoint. In particular, we will investigate the connection between the ergodic theoretical properties of the tail measure and those of the probability law of the process \( X \).

Here we need to recall the so-called positive-null decomposition by which the ergodic properties of the tail measure will be rigorously described. For details, we refer to Wang et al. (2011), which is essentially based on Takahashi (1971). See also Aaronson (1997) and Krengel (1985). First, suppose that a tail measure \( \nu \) is \( \sigma \)-finite (a necessary and sufficient condition for \( \sigma \)-finiteness is given in Proposition 2.4). Then \( (\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T), \nu) \) becomes a standard Lebesgue measure space (see Appendix A in Pipiras and Taqqu (2004) for the terminology). We define

\[
\Lambda = \{ Q \ll \nu : Q \text{ is a finite measure on } \mathbb{R}^T, Q \circ \phi_t^{-1} = Q \text{ for all } t \in T \} ,
\]

\[
S_Q = \{ x \in \mathbb{R}^T : \frac{dQ}{d\nu}(x) > 0 \} , \quad Q \in \Lambda .
\]

According to Lemma 2.2 in Wang et al. (2011), \( \{ S_Q : Q \in \Lambda \} \) has a unique maximal element \( P \) in the sense that

(i): for all \( R \in \Lambda \), \( \nu(S_R \setminus P) = 0 \),

(ii): if there exists another \( P' \) satisfying (i), then \( P = P' \mod \nu \).

Then, such \( P \) is called a positive part and \( N = \mathbb{R}^T \setminus P \) is a null part. It is shown by Theorem 2.3 in Wang et al. (2011) that both \( P \) and \( N \) are invariant with respect to \( (\phi_t, t \in T) \), i.e., for all \( t \in T \),

\[
\mu(\phi_t^{-1}(P) \Delta P) = 0 \quad \text{and} \quad \mu(\phi_t^{-1}(N) \Delta N) = 0 .
\]

If \( \mathbb{R}^T = P \mod \nu \), then \( (\phi_t, t \in T) \) is said to be a positive flow, and if \( \mathbb{R}^T = N \mod \nu \), then it is called a null flow.

Our first result below relates the ergodic properties of the flow \( (\phi_t, t \in T) \) defined on \( (\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T), \nu) \) to the Cesàro type convergence of \( \nu \{ x \in \mathbb{R}^T : |x_0| > 1, \ |x_t| > 1 \} \). More precisely, if \( (\phi_t, t \in T) \) is a null flow, \( \nu \{ x \in \mathbb{R}^T : |x_0| > 1, \ |x_t| > 1 \} \) converges to zero in the Cesàro sense. On the contrary, if \( (\phi_t, t \in T) \) has a positive component, then the same quantity does not converge to zero in the Cesàro sense. Alternatively, we may say that if \( (\phi_t, t \in T) \) has a positive component, then the original process \( X \) exhibits stronger dependence among their extremes.

**Proposition 5.1.** Let \( \lambda \) denote either the counting measure (if \( T = \mathbb{Z} \)) or the Lebesgue measure (if \( T = \mathbb{R} \)).

(i): If \( (\phi_t, t \in T) \) is a null flow on \( (\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T), \nu) \), then

\[
\frac{1}{T} \int_{[0, T]} \nu \{ x \in \mathbb{R}^T : |x_0| > \delta, \ |x_t| > \delta \} \lambda(dt) \to 0
\]
for every $\delta > 0$.

(ii): If $(\phi_t, t \in T)$ has a positive component on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T, \nu)$, then

$$\liminf_{T \to \infty} \frac{1}{T} \int_{[0,T]} \nu\left\{ x \in \mathbb{R}^T : |x_0| > \delta, |x_t| > \delta \right\} \lambda(dt) > 0$$

for every $\delta > 0$ with $\nu\{x \in P : |x_0| > \delta \} > 0$. Here, $P$ is a positive part of $\mathbb{R}^T$.

**Proof.** Because of an invariance property of $\nu$, it suffices to check these statements when $\delta = 1$.

(i): Let $(\phi_t, t \in T)$ be a null flow defined on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T, \nu)$. Then,

$$\frac{1}{T} \int_{[0,T]} \nu\{ x \in \mathbb{R}^T : |x_0| > 1, |x_t| > 1 \} \lambda(dt) = \int_0^1 \nu \left\{ x \in A : \frac{1}{T} \int_{[0,T]} 1_A \circ \phi_t(x) \lambda(dt) > y \right\} dy,$$

where $A = \{ x \in \mathbb{R}^T : |x_0| > 1 \}$ is a measurable set of $\nu$-finite measure. It follows from Krengel’s stochastic ergodic theorem (see Theorem 4.9 of Krengel (1985)) that

$$\nu \left\{ x \in A : \frac{1}{T} \int_{[0,T]} 1_A \circ \phi_t(x) \lambda(dt) > y \right\} \to 0$$

for every $0 \leq y \leq 1$. So, the result follows.

(ii): By virtue of (i), we may assume without loss of generality that $(\phi_t, t \in T)$ is a positive flow on the whole measure space $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T, \nu)$. Then, there exists a probability measure $Q$ that is equivalent to $\nu$ and is preserved under $(\phi_t, t \in T)$. Let $g = dQ/d\nu$ be its Radon-Nikodym derivative. Let $A = \{ x \in \mathbb{R}^T : |x_0| > 1 \}$. Since $Q$ is a probability measure, the Birkhoff ergodic theorem yields

$$\frac{1}{T} \int_{[0,T]} 1_A \circ \phi_t(x) \lambda(dt) \to E_Q(1_A|\mathcal{I}), \quad Q\text{-a.e.,}$$

where $\mathcal{I}$ is the $\sigma$-field of all $(\phi_t, t \in T)$-invariant measurable sets.

Consequently,

$$\frac{1}{T} \int_{[0,T]} \int_{A \cap \phi_t^{-1}A} g(x) \nu(dx) \lambda(dt) = \int_A \frac{1}{T} \int_{[0,T]} 1_A \circ \phi_t(x) \lambda(dt) Q(dx) \to \int_A E_Q(1_A|\mathcal{I}) dQ > 0.$$

Choose $K > 0$ so that

$$\int_A g(x) 1_{(g(x) > K)} \nu(dx) \leq \frac{1}{2} \int_A E_Q(1_A|\mathcal{I}) dQ.$$

Now, we have

$$\frac{1}{T} \int_{[0,T]} \int_{A \cap \phi_t^{-1}A} g(x) \nu(dx) \lambda(dt) \leq \frac{1}{2} \int_A E_Q(1_A|\mathcal{I}) dQ + \frac{K}{T} \int_{[0,T]} \nu(A \cap \phi_t^{-1}A) \lambda(dt).$$

Therefore,

$$\liminf_{T \to \infty} \frac{1}{T} \int_{[0,T]} \nu(A \cap \phi_t^{-1}A) \lambda(dt) \geq \frac{1}{2K} \int_A E_Q(1_A|\mathcal{I}) dQ > 0.$$
In the sequel, we only focus on the process studied in Examples 3.1 and 3.2: with $T = \mathbb{Z}$ or $\mathbb{R}$,
\begin{equation}
X_t = \int_E f_t(x) dM(x), \quad t \in T.
\end{equation}
Here $M$ is an independently scattered infinitely divisible random measure on a measurable space $(E, \mathcal{E})$ with local Lévy measure $\rho(s, \cdot)$, $s \in E$, and $\sigma$-finite control measure $m$. Assume that $M$ has no Gaussian component. In other words, the characteristic function of $M(A)$, for $m$-finite set $A \in \mathcal{E}$, is given by
\begin{equation}
E e^{iuM(A)} = \exp \left[ \int_A \left\{ iub + \int_{\mathbb{R}} (e^{iux} - 1 - iux) \rho(s, dx) \right\} m(ds) \right],
\end{equation}
where $b : E \to \mathbb{R}$ and $\tau(x) = x/\max\{1, |x|\}$. The functions $f_t$ are defined by
\begin{equation}
f_t(x) = f \circ \psi_t(x), \quad x \in E, \quad t \in T,
\end{equation}
where $\psi_t : E \to E$, $t \in T$, is a family of measurable maps, and $f : E \to \mathbb{R}$ is a measurable function. $\psi_t$ and $f$ are taken in such a way that the resulting process $X = (X_t, t \in T)$ becomes a stationary and well-defined infinitely divisible process; see Rajput and Rosiński (1989).

Examples 3.1 and 3.2 both have proved that the tail measure of $X$ is $(\rho_s \times m) \circ h^{-1}$, where $\rho_s(dx)$ is defined by either (3.5) or (3.10). As seen in Proposition 5.1, the Cesàro convergence of $(\rho_s \times m) \circ h^{-1}\{ x \in \mathbb{R}^T : |x_0| > \delta, \ |x_t| > \delta \}$ is characterized by the ergodic theoretical properties of the flow $(\phi_t, t \in T)$ defined on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R})^T)$. Moreover, ergodicity of the probability law of $X$ is characterized by the Cesàro convergence of the Lévy measure of $X$. Namely, $X$ is ergodic if and only if, for every $\eta > 0$,
\begin{equation}
\frac{1}{T} \int_{[0,T]} (\rho \times m) \circ h^{-1}\{ x \in \mathbb{R}^T : |x_0| > \eta, \ |x_t| > \eta \} \lambda(dt) \to 0 \quad \text{as } T \to \infty.
\end{equation}
See e.g. Rosiński and Žak (1997). Due to the similarity of the Lévy measure $(\rho \times m) \circ h^{-1}$ and the tail measure $(\rho_s \times m) \circ h^{-1}$, strong connection between the ergodic properties of $(\phi_t, t \in T)$ and ergodicity of $X$ is expected to exist.

**Theorem 5.2.** Let $(X_t, t \in T)$ be a stationary infinitely divisible process of the form (5.1), where $M$ is an independently scattered infinitely divisible random measure given in (5.2), and $f_t$ is defined in (5.3). We assume (3.2) and (3.4) and, furthermore, a regularly varying function $H : (0, \infty) \to (0, \infty)$ is bounded away from infinity on every compact interval. We assume a stronger version of (3.3): that is, for all $v > 0$, there exists a $K(v) > 0$, such that
\begin{equation}
\sup_{u \geq v} \frac{\rho(s, (u, \infty))}{H(u)} \leq K(v)w_+(s) \quad \text{and} \quad \sup_{u \geq v} \frac{\rho(s, (-\infty, -u))}{H(u)} \leq K(v)w_-(s)
\end{equation}
for all \( s \in E \). We put an extra assumption on the lower bound of the quantities in (3.2): there exists \( u_0 > 0 \) and \( L > 0 \), such that

\[
(5.5) \quad \frac{\rho(s, (u_0, \infty))}{H(u_0)} \geq Lw_+(s) \quad \text{and} \quad \frac{\rho(s, (-\infty, -u_0))}{H(u_0)} \geq Lw_-(s)
\]

for all \( s \in E \).

Applying the positive-null decomposition to the tail measure \( \nu = (\rho_s \times m) \circ h^{-1} \), we have \( \nu = \nu|_N + \nu|_P \). Then \( (X_t, t \in T) \) is ergodic if and only if \( \nu|_P \) is identically zero.

Proof. Recall that \( (X_t, t \in T) \) is ergodic if and only if, for every \( \eta > 0 \),

\[
(5.6) \quad \frac{1}{T} \int_{[0,T]} (\rho \times m)\{(x,s) : |xf(s)| > \eta, |xf_t(s)| > \eta\} \lambda(dt) \to 0 \quad \text{as} \quad T \to \infty.
\]

First, we will prove that (5.6) is equivalent to

\[
(5.7) \quad \frac{1}{T} \int_{[0,T]} \int_{A_t^{(\epsilon)}} (w_+(s) + w_-(s))m(ds)\lambda(dt) \to 0 \quad \text{as} \quad T \to \infty.
\]

for every \( \epsilon > 0 \), where \( A_t^{(\epsilon)} = \{ s \in E : |f(s)| > \epsilon, |f_t(s)| > \epsilon \} \).

Assume that (5.6) holds for every \( \eta > 0 \). For any \( \epsilon > 0 \), let \( \delta = \epsilon u_0 \). Then

\[
\frac{1}{T} \int_{[0,T]} (\rho \times m)\{(x,s) : |xf(s)| > \delta, |xf_t(s)| > \delta\} \lambda(dt)
\]

\[
\geq \frac{1}{T} \int_{[0,T]} (\rho \times m)\{(x,s) : |x| > u_0, |f(s)| > \epsilon, |f_t(s)| > \epsilon\} \lambda(dt)
\]

\[
= \frac{H(u_0)}{T} \int_{[0,T]} \int_{A_t^{(\epsilon)}} \frac{\rho(s, \{x : |x| > u_0\})}{H(u_0)} m(ds) \lambda(dt)
\]

\[
\geq \frac{LH(u_0)}{T} \int_{[0,T]} \int_{A_t^{(\epsilon)}} (w_+(s) + w_-(s)) m(ds) \lambda(dt).
\]

Here, the last inequality follows from (5.5) and, thus, (5.6) completes one direction of the assertion.

Conversely, assume that (5.7) holds for any \( \eta > 0 \), we split the integral in (5.6) into three parts.

\[
\frac{1}{T} \int_{[0,T]} (\rho \times m)\{(x,s) : |xf(s)| > \eta, |xf_t(s)| > \eta\} \lambda(dt)
\]

\[
= \frac{1}{T} \int_{[0,T]} \int_{|f(s)| \leq \delta} \rho(s, \{x : |xf(s)| > \eta, |xf_t(s)| > \eta\}) m(ds) \lambda(dt)
\]

\[
+ \frac{1}{T} \int_{[0,T]} \int_{|f(s)| > \delta, |f_t(s)| \leq \epsilon} \rho(s, \{x : |xf(s)| > \eta, |xf_t(s)| > \eta\}) m(ds) \lambda(dt)
\]

\[
+ \frac{1}{T} \int_{[0,T]} \int_{|f(s)| > \delta, |f_t(s)| > \epsilon} \rho(s, \{x : |xf(s)| > \eta, |xf_t(s)| > \eta\}) m(ds) \lambda(dt)
\]

\[
= I_1 + I_2 + I_3.
\]
Notice that \((\rho \times m)\{(x, s) : |xf(s)| > \eta\} < \infty\), since the process \(X\) is well-defined. For the first term \(I_1\), the stationarity of the process and the Cauchy-Schwarz inequality give the upper bound

\[
I_1 \leq (\rho \times m)\{(x, s) : |xf(s)| > \eta, |f(s)| \leq \delta\}^{1/2}(\rho \times m)\{(x, s) : |xf(s)| > \eta\}^{1/2}.
\]

The right hand side above converges to zero as \(\delta \downarrow 0\), by the dominated convergence theorem. Next, we get

\[
I_2 \leq (\rho \times m)\{(x, s) : |xf(s)| > \eta, |x| > \eta/\epsilon\},
\]

which goes to zero as \(\epsilon \downarrow 0\) by the dominated convergence theorem.

Fix \(\delta > 0\) and \(\epsilon > 0\) so small that both \(I_1\) and \(I_2\) are sufficiently small. Applying the Cauchy-Schwarz inequality,

\[
I_3 \leq \left( \frac{1}{T} \int_{[0,T]} \int_{|f(s)| > \delta, |f_t(s)| > \epsilon} \rho\left(s, \{x : |xf(s)| > \eta\}\right) m(ds)\lambda(dt) \right)^{1/2} (\rho \times m)\{(x, s) : |xf(s)| > \eta\}^{1/2}.
\]

Thus, it suffices to show that, for every \(\epsilon > 0\),

\[
\frac{1}{T} \int_{[0,T]} \int_{A_t^{(s)}} \rho\left(s, \{x : |xf(s)| > \eta\}\right) m(ds)\lambda(dt) \to 0 \quad \text{as} \quad T \to \infty.
\]

From (5.4), we have

\[
\frac{1}{T} \int_{[0,T]} \int_{A_t^{(s)}} \rho\left(s, \{x : |xf(s)| > \eta\}\right) m(ds)\lambda(dt) \leq \frac{K(\eta/\sup_{s \in E}|f(s)|)}{T} \int_{[0,T]} \int_{A_t^{(s)}} (w_+(s) + w_-(s)) H(\eta|f(s)|^{-1}) m(ds)\lambda(dt).
\]

Since \(H\) is bounded away from infinity on every compact interval, the Potter bounds yields, for some \(C > 0\),

\[
H(\eta|f(s)|^{-1}) \leq C \left( \frac{\eta}{\sup_{s \in E}|f(s)|} \right)^{-\alpha/2} < \infty
\]

for all \(s \in E\). Therefore, (5.7) completes the other direction of the assertion.

Now we have checked that (5.6) and (5.7) are equivalent. Observe that even if one replaces \(\rho\) with \(\rho_\ast\) defined in (3.5), statements (5.6) and (5.7) are still equivalent. In fact, \(\rho_\ast\) satisfies (3.2), (4.5) and (5.5), if we set \(H(u) = u^{-\alpha}\). In conclusion, (5.6) is equivalent to

\[
\frac{1}{T} \int_{[0,T]} (\rho_\ast \times m)\{(x, s) : |xf(s)| > \eta, |xf\}_t(s)| > \eta\} \lambda(dt) \to 0 \quad \text{as} \quad T \to \infty
\]

for every \(\eta > 0\). However, we find from Proposition 5.1 that (5.8) holds if and only if \(\nu|_\rho\) is identically zero.

We will, next, study the process given in Example 3.2.

**Theorem 5.3.** Let \((X_t, t \in T)\) be a stationary infinitely divisible process of the form (5.1), where \(M\) is an independently scattered infinitely divisible random measure given in (5.2), and \(f_t\) is defined...
in (5.3). However, we let \( \rho \) be independent of \( s \in E \). We assume tail balanced regularly varying condition (3.9). We will specify the integrability of \( f \) as follows: for every \( t \in T \),
\[
\int_E |f_t(s)|^{\alpha - \xi} \vee |f_t(s)|^2 m(ds) < \infty \quad \text{for some } 0 < \xi < 2 - \alpha \quad \text{if } 0 < \alpha < 2, \\
\int_E |f_t(s)|^{\alpha - \xi} \vee |f_t(s)|^{\alpha + \xi} m(ds) < \infty \quad \text{for some } 0 < \xi < \alpha \quad \text{if } \alpha \geq 2.
\]
Furthermore, if \( 0 < \alpha < 2 \), the lower tail of \( \rho \) is assumed to satisfy
\[
(5.9) \quad x^{p_0} \rho(y : |y| > x) \to 0 \quad \text{as } x \downarrow 0
\]
for some \( p_0 \in (\alpha, 2) \).

Under this setup, \((X_t, t \in T)\) is ergodic if and only if \( \nu|_P \) is identically zero.

**Proof.** We only prove that
\[
(5.10) \quad \frac{1}{T} \int_{[0,T]} (\rho \times m) \{ (x, s) : |xf(s)| > \eta, \ |xf_t(s)| > \eta \} \lambda(dt) \to 0, \quad \text{for every } \eta > 0,
\]
is equivalent to
\[
(5.11) \quad \frac{1}{T} \int_{[0,T]} m(A^{(\epsilon)}_t) \lambda(dt) \to 0 \quad \text{for every } \epsilon > 0,
\]
where \( A_t^{(\epsilon)} = \{ s \in E : |f(s)| > \epsilon, \ |f_t(s)| > \epsilon \} \). Once the above equivalence is established, the rest of the argument is almost the same as that in Theorem 5.2.

First, we assume (5.10). For any \( \epsilon > 0 \),
\[
\frac{1}{T} \int_{[0,T]} (\rho \times m) \{ (x, s) : |xf(s)| > \epsilon, \ |xf_t(s)| > \epsilon \} \lambda(dt)
\]
\[
\geq \frac{1}{T} \int_{[0,T]} (\rho \times m) \{ (x, s) : |x| > 1, \ |f(s)| > \epsilon, \ |f_t(s)| > \epsilon \} \lambda(dt)
\]
\[
= \frac{\rho \{ x : |x| > 1 \}}{T} \int_{[0,T]} m(A^{(\epsilon)}_t) \lambda(dt).
\]
Thus, \( T^{-1} \int_{[0,T]} m(A^{(\epsilon)}_t) \lambda(dt) \to 0 \) as \( T \to \infty \).

Assume, conversely, that (5.11) holds. Once again, we need split the integral in (5.10) into three terms. For every \( \eta > 0 \),
\[
\frac{1}{T} \int_{[0,T]} (\rho \times m) \{ (x, s) : |xf(s)| > \eta, \ |xf_t(s)| > \eta \} \lambda(dt)
\]
\[
= \frac{1}{T} \int_{[0,T]} \int_{|f(s)| \leq \delta} \rho(x : |xf(s)| > \eta, \ |xf_t(s)| > \eta) m(ds) \lambda(dt)
\]
\[
+ \frac{1}{T} \int_{[0,T]} \int_{|f(s)| > \delta, \ |f_t(s)| \leq \epsilon} \rho(x : |xf(s)| > \eta, \ |xf_t(s)| > \eta) m(ds) \lambda(dt)
\]
\[
+ \frac{1}{T} \int_{[0,T]} \int_{|f(s)| > \delta, \ |f_t(s)| > \epsilon} \rho(x : |xf(s)| > \eta, \ |xf_t(s)| > \eta) m(ds) \lambda(dt)
\]
\[
= I_1 + I_2 + I_3.
\]
By a similar argument as the proof of Theorem 5.2, \( I_1 \) and \( I_2 \) can be arbitrarily small by taking \( \delta > 0 \) and \( \epsilon > 0 \) sufficiently small. Having fixed such \( \delta > 0 \) and \( \epsilon > 0 \) and assuming \( \epsilon < \delta \) without loss of generality, we have

\[
I_3 \leq \frac{1}{T} \int_{[0, T]} \int_E 1_{A_1^{(e)}}(s) \rho(x : |xf(s)| > \eta) m(ds) \lambda(dt).
\]

If \( 0 < \alpha < 2 \), an application of the Hölder’s inequality provides

\[
I_3 \leq \left( \frac{1}{T} \int_{[0, T]} m(A_1^{(e)}) \lambda(dt) \right)^{1-p_0/2} \left( \frac{1}{T} \int_{[0, T]} \int_{A_1^{(e)}} \rho(x : |xf(s)| > \eta)^{2/p_0} m(ds) \lambda(dt) \right)^{p_0/2}.
\]

By virtue of (5.11), it is enough to verify

\[
\limsup_{T \to \infty} \frac{1}{T} \int_{[0, T]} \int_{A_1^{(e)}} \rho(x : |xf(s)| > \eta)^{2/p_0} m(ds) \lambda(dt) < \infty.
\]

Indeed,

\[
\limsup_{T \to \infty} \frac{1}{T} \int_{[0, T]} \int_{A_1^{(e)}} \rho(x : |xf(s)| > \eta)^{2/p_0} m(ds) \lambda(dt)
\leq \int_E \rho(x : |x| > \eta |f(s)|^{-1})^{2/p_0} 1_{\{\eta \leq |f(s)|\}} m(ds).
\]

Because of (5.9),

\[
\rho(x : |x| > \eta |f(s)|^{-1}) 1_{\{\eta \leq |f(s)|\}} \leq C (\eta |f(s)|^{-1})^{-p_0}
\]

for some \( C > 0 \). Since \( f \in L^2(E) \), (5.12) follows.

In case of \( \alpha \geq 2 \), let \( \epsilon_0 \in (0, \xi) \). From the Hölder’s inequality,

\[
I_3 \leq \left( \frac{1}{T} \int_{[0, T]} m(A_1^{(e)}) \lambda(dt) \right)^{\epsilon_0/(\alpha + \xi)}
\times \left( \frac{1}{T} \int_{[0, T]} \int_{A_1^{(e)}} \rho(x : |xf(s)| > \eta)^{\alpha/\alpha + \xi} m(ds) \lambda(dt) \right)^{1 - \epsilon_0/(\alpha + \xi)}.
\]

In this case, we have, for some \( C > 0 \),

\[
\limsup_{T \to \infty} \frac{1}{T} \int_{[0, T]} \int_{A_1^{(e)}} \rho(x : |xf(s)| > \eta)^{(\alpha + \xi)/(\alpha + \xi - \epsilon_0)} m(ds) \lambda(dt)
\leq \int_E \rho(x : |x| > \eta |f(s)|^{-1})^{\alpha/\alpha + \xi} 1_{\{\eta \leq |f(s)|\}} m(ds)
\leq C \eta^{-(\alpha + \xi)} \int_E |f(s)|^{\alpha + \xi} m(ds) < \infty.
\]

The last inequality follows from \( y^{\alpha + \xi - \epsilon_0} \rho(x : |x| > y) \to 0 \) as \( y \downarrow 0 \). Now, in either case, \( \limsup_{T \to \infty} I_3 = 0 \) and, hence, (5.10) has been established.
References


